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Observables and Transformation Properties of Fermions and Parafermions Constructed in Terms of Bosons. II. Transformation Laws

ANDRÉS J. KÁLNAY

Centro de Fisica, IVIC, Apartado 1827, *Caracas* 101, *Venezuela*

and K. V. KADEMOVA†

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With the results and notations of Part I* and Kálnay, 1974 (hereafter called B),‡ we demonstrate what follows. Let $\mathscr{P}$ the Poincaré transformation $x_\mu = \Sigma_\nu l_{\mu\nu} x_\nu + a_\mu$. Let

$$U^{\mathscr{B}}(b, b^+) b_\xi(x) [U^{\mathscr{B}} b, b^+)]^{-1} = \sum_{\hat{\xi}} (\Lambda^{-1})_{\xi\hat{\xi}} b_{\hat{\xi}}(\hat{x}) \tag{1}$$

$$U_{\mathscr{F}}(f, f^+) f_\xi(z) [U_{\mathscr{F}}(f, f^+)]^{-1} = \sum_{\hat{\xi}} (\Lambda^{-1})_{\xi\hat{\xi}} f_{\hat{\xi}}(\hat{z}) \tag{2}$$

be the unitary induced transformations on the Bose and Fermi (para-Fermi) fields b and f of the standard theory, i.e. the one in which the f fields are primary entities, that is, non-Bose-constructed. We want to have the same transformation law (2) for f in the Bose representation of those fields in which the unitary transformations are Bose constructed operators $U_{\mathscr{B}}(b, b^+)$:

$$U_{\mathscr{B}}(b, b^+) f_\xi(z) [U_{\mathscr{B}}(b, b^+)]^{-1} = \sum_{\hat{\xi}} (\Lambda^{-1})_{\xi\hat{\xi}} f_{\hat{\xi}}(z) \tag{3}$$

† See footnote of Part I which applies also to this paper.

* Kálnay & Kademova (1974).

‡ We follow the same abbreviations for the references as those of Part I, hereafter called I.

Remark 1. Because of Stone's theorem, there exists Hermitian operators $G^{\mathscr{B}}(b, b^+)$, $G_{\mathscr{F}}(f, f^+)$ and $G_{\mathscr{B}}(b, b^+)$ such that

$$U^{\mathscr{B}} = \exp iG^{\mathscr{B}}, \qquad U_{\mathscr{F}} = \exp iG_{\mathscr{F}}, \qquad U_{\mathscr{B}} = \exp iG_{\mathscr{B}} \tag{4}$$

Theorem 2. To have in the Bose representation of Fermi and para-Fermi fields the transformation law (3), put

$$U_{\mathscr{B}}(b, b^+) = U_{\mathscr{F}}[f(b, b^+), f^+(b, b^+)] \tag{5}$$

Proof. Use Remark 1 and the Theorem of I. □

Remark 3. The same reasoning leads to equation (5) when it is induced by a non-spatial transformation as, for example, gauge transformations.

Remark 4. Let us consider the case when the G's are (up to c-number parameters) the angular momenta $M^{\mu\nu}$. Because of I, the algebraic relations among $f(b, b^+)$ and $M^{\mu\nu}_{\mathscr{B}}(b, b^+)$ are the same as those among f and $(M^{\mathscr{F}})^{\mu\nu}$ of the standard theory. But it is known that these last relations determine the spinor transformation properties of f. Therefore, the Bose constructed $f(b, b^+)$ also transforms as a spinor. In order to visualise that this does not contradict the tensor transformation properties of b we shall develop the formalism further. In what follows we assume the conditions for Fock representation for the Fermi or para-Fermi commutation rules (cf. B).

Definition 5. We put $z = (z_0, \mathbf{z})$, $x = (x_0, x)$, $x' = (x_0, x')$. Then we call

$$F_{\xi\zeta\zeta'}(z, x, x') = {}^{\mathscr{B}}\langle 0 \,|\, b_\zeta(x) f_\xi(z) b^+_{\zeta'}(x') \,|\, 0 \rangle^{\mathscr{B}} \tag{6}$$

Theorem 6. When $x_0 = x_0'$ the $F_{\xi\zeta\zeta'}(z, x, x')$ are matrix elements of a z_0 time-dependent Fermi matrix $F_\xi(z)$.

Proof. Taking into account equations (3) and (4) of I, put $f_\xi'(z) = Vf_\xi(z)V^{-1}$, $V = \exp(-iHx_0)\exp(iHz_0)$. As f is Fermi or para-Fermi, the same applies to f'. Then by Theorem 3.2 of B we prove that $F_\xi(z)$ is Fermi. Take into account equations (3)–(4) of Part I. □

Theorem 7. Under $\mathscr{P}$ the $F_{\xi\zeta\zeta'}(z, x, x')$ transforms according to

$$F_{\hat\xi\hat\zeta\hat\zeta'}(\hat z, \hat x, \hat x') = \sum_{\xi\zeta\zeta'} \Lambda_{\hat\xi\xi}\lambda_{\hat\zeta\zeta}\lambda_{\hat\zeta'\zeta'}F_{\xi\zeta\zeta'}(z, x, x') \tag{7}$$

Proof. Use equations (1), (3) and (6). □

Theorem 8. The manifestly covariant form of the Bose representation of Fermi or para-Fermi fields is

$$f_\xi(z) = \sum_{\zeta\zeta'} \int_\sigma d\sigma \int_\sigma d\sigma' F_{\xi\zeta\zeta'}(z, x, x') b_\zeta^+(z) b_{\zeta'}(x') \tag{8}$$

where σ is an arbitrary space-like hypersurface.

Proof. The covariance is guaranteed by equations (1), (3) and (7). In order to prove that the right-hand side of (8) is a Fermi operator when acting on

$\mathscr{B}_1$ (or a para-Fermi when acting on $\mathscr{B}_p$) it is sufficient to construct the proof in a special frame; choose an instantaneous σ to be defined by $x_0 = z_0$. Then the right-hand side of (8) reduces to the Bose representation of fermions (respectively parafermions) given in papers A (Kálnay, MacCotrina & Kademova, 1974) and B. Finally, the proof that the evolution of both sides of equation (8) according to the time z_0 is the same can be achieved in a frame such that σ be instantaneous corresponding to $x_0 = 0$, but unrelated to z_0. Then, equation (8) is consistent as regards time evolution if given a $H_{\mathscr{B}}$, i.e. once fixed the z_0 dependence of $f_\xi(z)$, there exist $F_{\xi\zeta\zeta'}(z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}')$ and $F_{\xi\zeta\zeta'}(z_0 + \delta z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}')$ such that

$$f_\xi(z_0, \mathbf{z}) = \sum_{\zeta\zeta'} \int d^3x \int d^3x' F_{\xi\zeta\zeta'}(z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}') b_\zeta^+(\mathbf{x}) b_{\zeta'}(\mathbf{x}') \tag{9}$$

and

$$f_\xi(z_0 + \delta z_0, \mathbf{z}) = \sum_{\zeta\zeta'} \int d^3x \int d^3x' F_{\xi\zeta\zeta'}(z_0 + \delta z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}') b_\zeta^+(\mathbf{x}) b_{\zeta'}(\mathbf{x}') \tag{10}$$

But according to Theorem 3.2 of B, such F exist and can be made explicit as

$$F_{\xi\zeta\zeta'}(z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}') = {}^{\mathscr{B}}\langle 0 | b_\zeta(\mathbf{x}) f_\xi(z_0, \mathbf{z}) b_{\zeta}^+(\mathbf{x}') | 0 \rangle^{\mathscr{B}} \tag{11}$$

and, by using equation (3) of I to complete f at $z_0 + \delta z_0$

$$F_{\xi\zeta\zeta'}(z_0 + \delta z_0, \mathbf{z}; \mathbf{x}, \mathbf{x}') = {}^{\mathscr{B}}\langle 0 | b_\zeta(\mathbf{x}) f_\xi(z_0 + \delta z_0, \mathbf{z}) b_{\zeta'}^+(\mathbf{x}') | 0 \rangle^{\mathscr{B}} \tag{12}$$

Remark 9. From the role in equation (8) of the tensor indices ζ, ζ' and of the spinor index ξ, it is apparent that *the Bose representation does not introduce conflict between the tensor transformation laws of bosons and the spinor laws of fermions and parafermions.*

Note added in proof: For the form that replaces the right-hand side of equation (8) when gauge constraints are present, see Kálnay, A. J., *Electrons and Photons in the Bose Description of Fermions,* Preprint IC/74/109, International Centre for Theoretical Physics, Trieste.

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